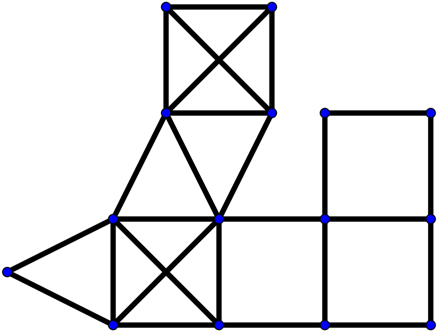
Elimination depth, a generalization of treedepth

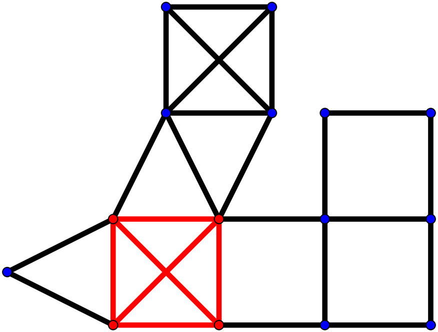
Thomas Delépine, Hoang La, François Pirot

Novembre 2024

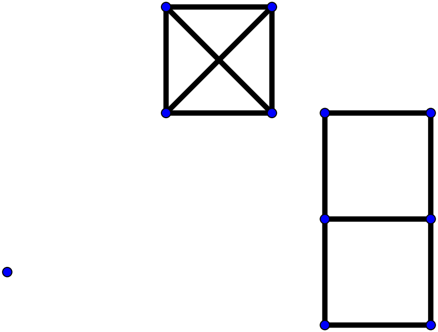
Elimination depth



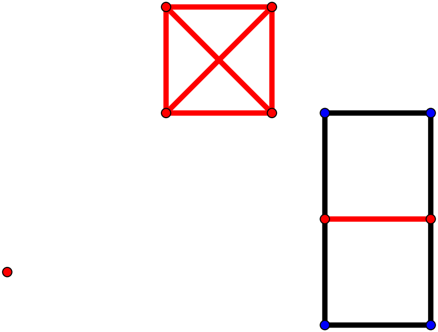
Elimination depth



Elimination depth



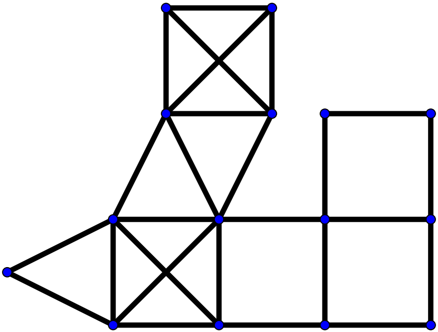
Elimination depth



Elimination depth

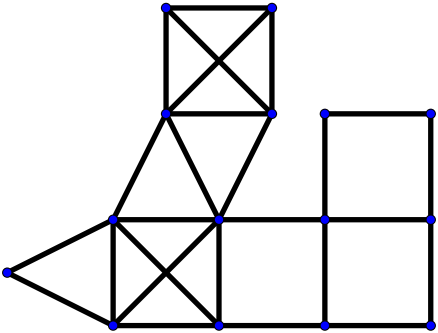
Elimination depth

Elimination depth



Graph eliminated in 3 rounds

Elimination depth



Definition

Let G be a graph,

$$\mathrm{ed}^{\mathcal{K}}(G) = \begin{cases} 0 & \text{if } V(G) \text{ is empty,} \\ \max_{\{H \text{ component of } G\}} \mathrm{ed}^{\mathcal{K}}(H) & \text{if } G \text{ is disconnected,} \\ \min_{\{H \subseteq G \,:\, H \in \mathcal{K}\}} 1 + \mathrm{ed}^{\mathcal{K}}(G - H) & \text{if } G \text{ is connected.} \end{cases}$$

Elimination depth

Definition

Let G be a graph, let $\mathcal{D}$ be a set of connected graphs containing K_1,

$$\mathrm{ed}^{\mathcal{D}}(G) = \begin{cases} 0 & \text{if } V(G) \text{ is empty,} \\ \max_{\{H \text{ component of } G\}} \mathrm{ed}^{\mathcal{D}}(H) & \text{if } G \text{ is disconnected,} \\ \min_{\{H \subseteq G \ : \ H \in \mathcal{D}\}} 1 + \mathrm{ed}^{\mathcal{D}}(G - H) & \text{if } G \text{ is connected.} \end{cases}$$

Elimination depth

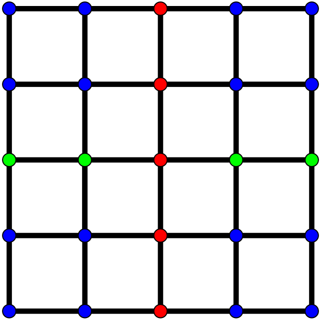
Definition

Let G be a graph, let $\mathcal{D}$ be a set of connected graphs containing K_1,

$$\mathrm{ed}^{\mathcal{D}}(G) = \begin{cases} 0 & \text{if } V(G) \text{ is empty,} \\ \max_{\{H \text{ component of } G\}} \mathrm{ed}^{\mathcal{D}}(H) & \text{if } G \text{ is disconnected,} \\ \min_{\{H \subseteq G \,:\, H \in \mathcal{D}\}} 1 + \mathrm{ed}^{\mathcal{D}}(G - H) & \text{if } G \text{ is connected.} \end{cases}$$

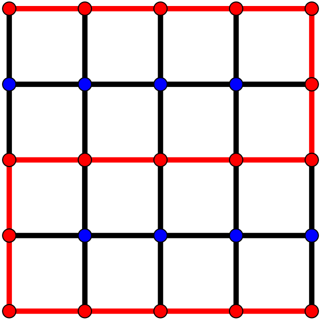
If $\mathcal{D} = \{K_1\}$, $\mathrm{ed}^{\mathcal{D}} = \mathrm{td}$

Un autre exemple



$$\mathrm{td}(G_{n,n}) = \Theta(n)$$

Un autre exemple

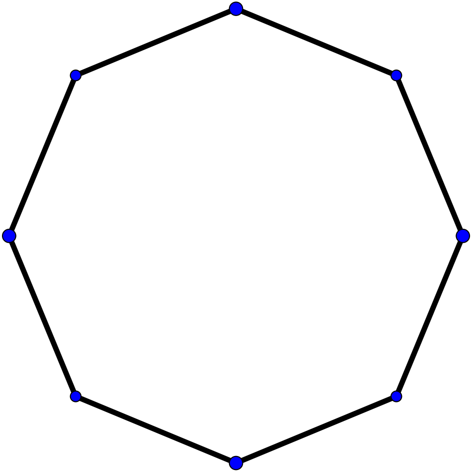


$$\mathrm{ed}^{\mathcal{P}}(G_{n,n}) = 2$$

Monotonicity

$\mathcal{D}$ = cycles + K_1

G :

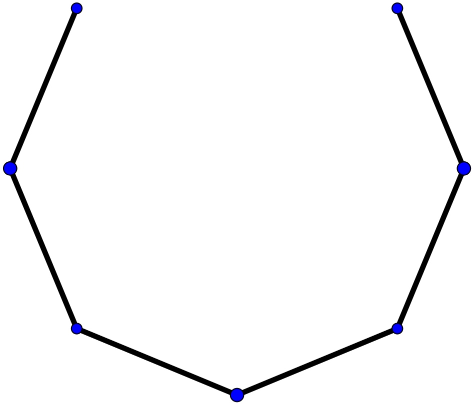


$\text{ed}^{\mathcal{D}}(G) = 1$

Monotonicity

$\mathcal{D} = \text{cycles} + K_1$

H :



$\text{ed}^{\mathcal{D}}(H) = \Theta(\log |V(H)|)$

Monotonicity

Theorem [D., La, Pirot, 2024+]

$\mathrm{ed}^{\mathcal{D}}$ is monotone w.r.t. the induced subgraph relation if and only if $\mathcal{D}$ is

- the set of all graphs,
- or the set of all cliques,
- or the set of all cliques of bounded size.

Monotonicity

Theorem [D., La, Pirot, 2024+]

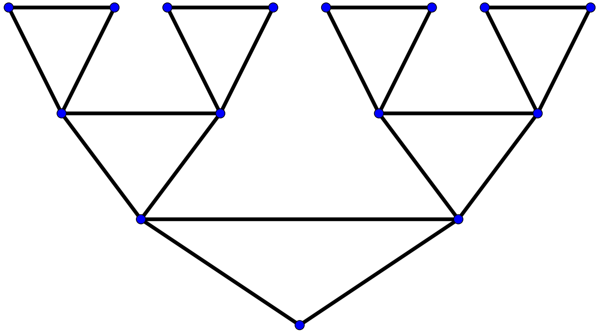
$\mathrm{ed}^{\mathcal{D}}$ is monotone w.r.t. the induced subgraph relation if and only if $\mathcal{D}$ is

- the set of all graphs,
- or the set of all cliques,
- or the set of all cliques of bounded size.

Question

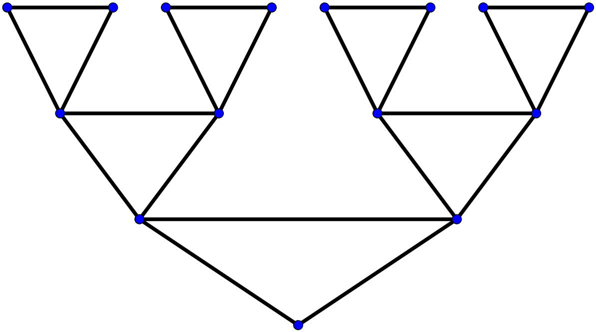
For fixed $\mathcal{D}$ and $\mathcal{C}$, how large can $\mathrm{ed}^{\mathcal{D}}(\mathcal{C})$ be ?

Delete trees in planar graphs



T_3

Delete trees in planar graphs



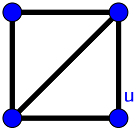
For $k \geq 1$, $\mathrm{ed}^{\text{trees}}(T_k) = \Theta(k) = \Theta(\log(n))$.

Theorem [D., La, Pirot, 2024+]

$\mathrm{ed}^{\text{trees}}(\text{planar graphs}) = \Omega(\log(n))$.

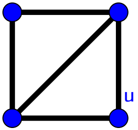
Logarithmic lower bound

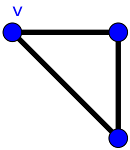
Dumbbell-operators : (G, u)



Logarithmic lower bound

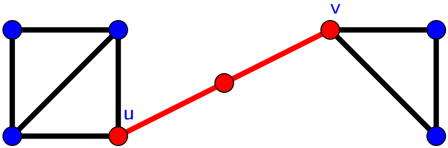
Dumbbell-operators : (G, u) (H, v)





Logarithmic lower bound

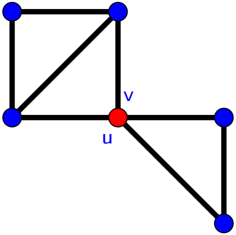
Dumbbell-operators : $(G, u) \bullet_p (H, v)$



$(G, u) \bullet_3 (H, v)$

Logarithmic lower bound

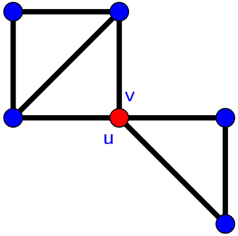
Dumbbell-operators : $(G, u) \bullet_p (H, v)$



$(G, u) \bullet_1 (H, v)$

Logarithmic lower bound

Dumbbell-operators : $(G, u) \bullet_p (H, v)$



$(G, u) \bullet_1 (H, v)$

Theorem [D., La, Pirot, 2024+]

Let $\mathcal{C}$ and $\mathcal{D}$ be sets of graphs such that $\mathcal{C} \not\subseteq \mathcal{D}$. If $\mathrm{ed}^{\mathcal{D}}$ is monotonic by dumbbell-operators and if there exists $p \geq 1$ such that $\mathcal{C}$ is closed by $\bullet_p$, then there exists $(G_k)_{k \geq 0} \in \mathcal{C}$ such that

$$\mathrm{ed}^{\mathcal{D}}(G_k) = \Omega(\log |V(G_k)|).$$

Logarithmic lower bound

Interior Property of S over $\bullet$: $a \bullet b \in S \implies a \in S$ and $b \in S$.

Logarithmic lower bound

Interior Property of S over $\bullet$: $a \bullet b \in S \implies a \in S$ and $b \in S$.

Proposition

Let $\mathcal{D}$ be a set of connected graphs containing K_1. $\mathcal{D}$ has the interior property over $\bullet_1$ if and only if for all graphs G, H and for all vertices $u \in V(G)$, $v \in V(H)$, and for all $p \geq 1$,

$$\mathrm{ed}^{\mathcal{D}}((G, u) \bullet_p (H, v)) \geq \max(\mathrm{ed}^{\mathcal{D}}(G), \mathrm{ed}^{\mathcal{D}}(G)).$$

Logarithmic lower bound

Interior Property of S over $\bullet$: $a \bullet b \in S \implies a \in S$ and $b \in S$.

Proposition

Let $\mathcal{D}$ be a set of connected graphs containing K_1. $\mathcal{D}$ has the interior property over $\bullet_1$ if and only if for all graphs G, H and for all vertices $u \in V(G)$, $v \in V(H)$, and for all $p \geq 1$,

$$\mathrm{ed}^{\mathcal{D}}((G,u) \bullet_p (H,v)) \geq \max(\mathrm{ed}^{\mathcal{D}}(G), \mathrm{ed}^{\mathcal{D}}(G)).$$

Theorem [D., La, Pirot, 2024+]

Let $\mathcal{C}$ and $\mathcal{D}$ be sets of graphs such that $\mathcal{C} \not\subseteq \mathcal{D}$. If $\mathcal{D}$ is a set of connected graph containing K_1, and with the interior property of $\bullet_1$ and if there exists $p \geq 1$ such that $\mathcal{C}$ is closed by $\bullet_p$, then there exists $(G_k)_{k \geq 0} \in \mathcal{C}$ such that

$$\mathrm{ed}^{\mathcal{D}}(G_k) = \Omega(\log |V(G_k)|).$$

Examples

$\mathcal{D}$ connected, contains K_1	interior property over $\bullet_1$
girth $\geq k+1$	C_k-minor free
tree-width $\leq k$	degeneracy $\leq k$
chromatic number $\leq k$	mad $\leq r$, $r > 2$
trees	cycles
cliques	$\{K_1\}$
planar graphs	eulerian graphs

$\mathcal{C}$ closed under $\bullet_p$	p
girth $\geq k$	1
tree-width $\leq k+1$	1
degeneracy $\leq k+1$	3
chromatic number $\leq k+1$	1
planar graphs	1
chordal graphs	1
K_t-minor free	1
C_{k+1}-minor free	1
mad $\leq r+\epsilon$, $r > 2$	$2+2/(r+\epsilon-2)$

Logarithmic upper bounds

Strategy

If $\mathcal{C}$ is hereditary and has a balanced separator that is easy to delete, then $\mathrm{ed}^{\mathcal{D}}(\mathcal{C}) = O(\log(n))$.

Logarithmic upper bounds

Strategy

If $\mathcal{C}$ is hereditary and has a balanced separator that is easy to delete, then $\mathrm{ed}^{\mathcal{D}}(\mathcal{C}) = O(\log(n))$.

- If G is 2-connected and C_{k+1}-minor free, then $\mathrm{ed}^{C_k\text{-minor free}}(G) \leq 2$. Elimination strategy: find a 2-connected balanced separator and delete it.

Logarithmic upper bounds

Strategy

If $\mathcal{C}$ is hereditary and has a balanced separator that is easy to delete, then $\mathrm{ed}^{\mathcal{D}}(\mathcal{C}) = O(\log(n))$.

- If G is 2-connected and C_{k+1}-minor free, then $\mathrm{ed}^{C_k\text{-minor free}}(G) \leq 2$. Elimination strategy: find a 2-connected balanced separator and delete it.

Theorem [Miller, 1986]

Let G be a 2-connected planar graph with an assignment of weights on its vertices. Then G contains an induced cycle that is a $2/3$-weighted-separator.

Logarithmic upper bounds

Strategy

If $\mathcal{C}$ is hereditary and has a balanced separator that is easy to delete, then $\mathrm{ed}^{\mathcal{D}}(\mathcal{C}) = O(\log(n))$.

- If G is 2-connected and C_{k+1}-minor free, then $\mathrm{ed}^{C_k\text{-minor free}}(G) \leq 2$. Elimination strategy: find a 2-connected balanced separator and delete it.

Theorem [Miller, 1986]

Let G be a 2-connected planar graph with an assignment of weights on its vertices. Then G contains an induced cycle that is a $2/3$-weighted-separator.

- Elimination strategy: find a 2-connected balanced separator, give clever weights and then delete the weighted separator of the 2-connected component.