Square-Free Arithmetic Progressions in Words

Journées SDA2 2025

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joint work with Pascal Ochem and Matthieu Rosenfeld

Words

- A word is a sequence $w = w_0 w_1 w_2 \ldots$ with $w_i \in \Sigma$
- w is a square if exists u, $w = uu$

  - Hotshots
  - Caracara
  - Chimachima

Theorem (Thue, 1906)

There exists infinite ternary square-free words.

Arithmetic progressions in square-free words

- For every $p \geq 2$, we denote $w_0 w_p w_{2p}\dots$ by $w[p]$

Theorem (Harju, 2019)

For every $p \geq 3$, there exists w an infinite ternary square-free word such that $w[p]$ is square-free.

Arithmetic progressions in square-free words

Question (Harju, 2019)

Do there exist pairs of relatively prime numbers p, q and an infinite ternary square-free word w such that $w[p]$ and $w[q]$ are square-free ?

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Yes for $(3, 11)$ and $(5, 6)$

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Question of interest

What are the "good" pairs ?

Large p and q

Morphisms : $h(0) = 01$, $h(1) = 12$, $h(2) = 20$

$$h(010) = 011201$$

h is uniform and circular

Large p and q

$$h_{23}(0) = 012102120210\mathbf{12}021201210$$
$$h_{24}(0) = 012102120210\mathbf{201}021201210$$
$$h_{25}(0) = 012102120210\mathbf{2012}021201210$$
$$h_{26}(0) = 012102120210\mathbf{20121}021201210$$

Proposition (Currie, Rampersad, Harju and Ochem, 2019)

h is a square-free multivalued morphism.

With h, we can do contractions !

$$\overbrace{1\bar{2}021\,01210212021020121021201210\,1\bar{2}02\hat{1}}^{h_{26}(0)}$$

310 341

$$\underbrace{1\bar{2}021\,0121021202101202120121 0\,1202\bar{1}}_{h_{23}(0)}$$

Large p and q

Strategy

If there exists a word w' such that $w'[p]$ and $w'[q]$ are square-free, then by using h, we can construct w square-free such that $w[p] = w'[p]$ and $w[q] = w'[q]$.

Large p and q

Strategy

If there exists a word w' such that $w'[p]$ and $w'[q]$ are square-free, then by using h, we can construct w square-free such that $w[p] = w'[p]$ and $w[q] = w'[q]$.

Let $(c_i)_{i \geq 0}$ be the positions of the forced letters.

Assume the word is correct up to position c_N, two cases :

1. if $c_{N+2} - c_{N+1} \geq 30$, then we treat c_{N+1} as a unique constraint, and we just need to contract one block
2. otherwise, we treat c_{N+1} and c_{N+2} as a pair of constraints, and we might need to contract a few blocks

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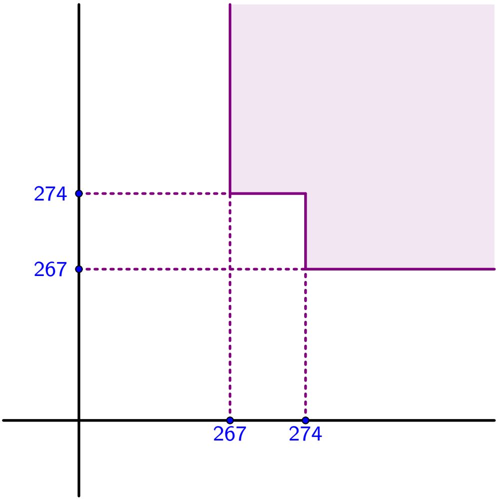
If p, $q \geq 267$, we can do it ! To construct w', we fix $w'[p]$ and proceed similarly, but we need $\max(p, q) \geq 274$.

Large p and q



Small p, large q

Theorem (Rosenfeld, 2020)

Let $(p_i)_{i\geq 0}$ be an increasing sequence of integers such that for every $i \geq 0$, $p_{i+1} - p_i \geq 19$ and let $(l_i)_{i\geq 0}$ be a sequence of letters. There exists an infinite ternary square-free word w such that for every $i \geq 0$, $w_{p_i} = l_i$.

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Construction procedure :

- Chose any infinite ternary square-free word, this will be $w[q]$.
- Assume that you have a morphism h that is square-free, p-uniform and circular (How lucky !)

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Theorem (Currie, 2012)

h exists for $p \in \{13, 17, 18, 19\} \cup \mathbb{N}_{\geq 23}$

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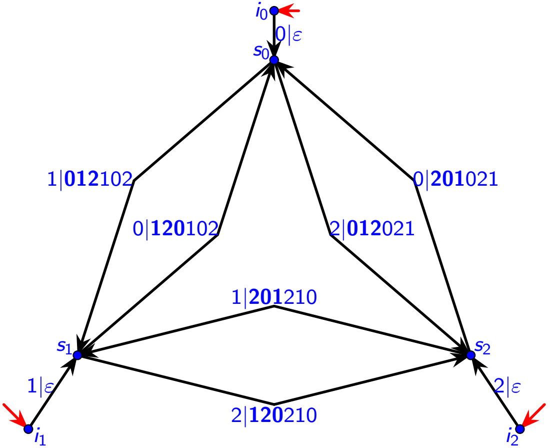
For the other values of p (3, 4, 5, 7, 8, 9, 10, 11, 12, 14, 15, 16, 20, 21, 22), we proceed almost similarly, but with larger morphism

$p = 6$, q large



i_0
$0|\varepsilon$
s_0
$1|\mathbf{012}102$
$0|\mathbf{120}102$
$2|\mathbf{012}021$
$0|\mathbf{201}021$
$1|\mathbf{201}210$
s_1
s_2
$1|\varepsilon$
$2|\varepsilon$
$2|\mathbf{120}210$
i_1
i_2

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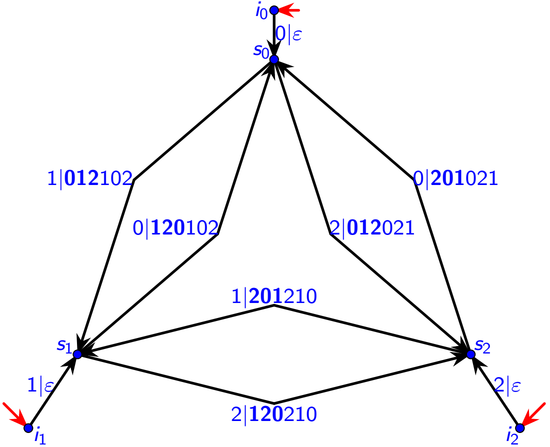
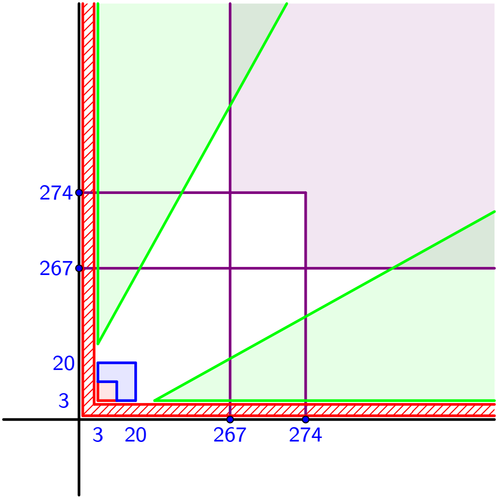
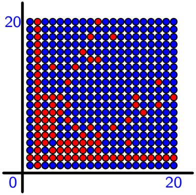


Proposition (D., Ochem, Rosenfeld, 2025)

For every $q \geq 384$, there exist an infinite ternary square-free word square-free modulo 6 and q

Set of good pairs





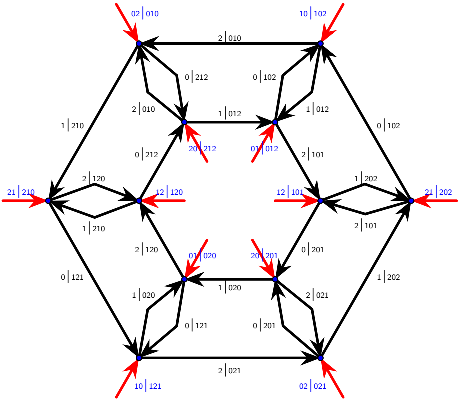
How many words ?

Exponential lower bounds

- In all our constructions, there is a step "chose your favorite infinite ternary square-free word" $\Rightarrow$ exponentially many words square-free, square-free mod p, square-free mod q.
- Let γ be such that $|\{\text{square-free words of size } n\}| = \Theta(\gamma^n)$ $(1.3017597 < \gamma < 1.3017619)$

For $p, q \geq 1000000$, this bound would be about $(1.3017^{1/1000000})^n$

How many words ? focus on word square-free modulo 3



Proposition (D., Ochem, Rosenfeld, 2025)

There are $\Theta((\gamma^{1/3})^n)$ words of size n that are square-free, square-free modulo 3.

Some Problems

- Fill the missing cases
- Natural Generalizations of this problem :
  - Variant of the problem on $\mathbb{N}^2$
  - Variant of the problem on k-uplets for $k \geq 3$
- Growth rate of the languages ?
- Is checking that a transducer is square-free decidable ?